Introduction to Valentin Turchin's Cybernetic Foundation of Mathematics

(Summary of Tutorial)

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Dedicated to the memory of V.F. Turchin (1931 - 2010) a thinker ahead of his time

Abstract. We introduce an alternative foundation of mathematics developed by Valentin Turchin in the 1980s — *Cybernetic Foundation of Mathematics.* This new philosophy of mathematics as human activity with formal linguistic machines is an extension of mathematical constructivism based on the notion of algorithms that define mathematical objects and notions as processes of a certain kind referred to as *metamechanical.* It is implied that all mathematical objects can be represented as metamechanical processes. We call this Turchin's thesis.

Keywords: constructive mathematics, alternative foundation of mathematics, Cybernetic Foundation of Mathematics, model of a mathematician, mechanical and metamechanical processes, objective interpretability, Turchin's thesis.

1 Introduction: Challenge of Foundation of Mathematics

Valentin Turchin's studies on the foundation of mathematics [1–4] comprise a highly underestimated part of his scientific legacy. Like some other great mathematicians of the 20th century, he was dissatisfied with the solution to the "crisis of mathematics" associated with the likes of David Hilbert and Nicolas Bourbaki - the formal axiomatic method. Its main drawback is that axioms, theorems and other formal texts of theories do not contain real mathematical objects. They refer to mathematical objects by means of variables and jargon, which are interpreted by human thought processes, but have no concrete representation in the stuff of language. Very few mathematical objects have names - constants like *true, false,* digits of numbers, *etc.* This corresponds to the Platonic philosophy of mathematics, an eternal immutable world of mathematical objects and perceived by minds and shaped in mathematical texts.

Valentin Turchin's take on mathematics is akin to *constructivism*: There is no Platonic world of mathematical objects. It is pure imagination. Mathematics is formal linguistic modeling of anything in the world including mathematics. Mathematical objects are abstract constructs represented by a formal language. The world of linguistic models is not static. Sentences in a formal language define processes that are potentially infinite

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sequences of states, where the states are also sentences in the language. These processes themselves are (representations of) mathematical objects. Mathematical objects are not a given, but are created by mathematicians in the process of their use of mathematical "machinery". This process of mathematical activity itself can be linguistically modeled, that is included into mathematics as well.

Turchin's thesis: Everything that humans may consider to be a mathematical object can be represented as a formal linguistic process. Valentin Turchin did not express this thesis explicitly. It is our interpretation of what we feel he implied.

The main problem is how to define a notion of linguistic processes to suit Turchin's thesis, that represents all objects mathematicians agree are mathematical. Valentin Turchin was not the first to imagine this kind of solution to the problem of foundation of mathematics. Attempts had been made by mathematicians in the early 20th century which failed. The idea was to equate the processes mentioned in the thesis with *algorithms*. Sophisticated mathematical theories were built on the basis of this idea in the 1950s and 1960s including *constructive mathematical analysis* that dealt with real numbers produced digit by digit by means of algorithms and functions that return their values digit by digit while gradually consuming digits of the arguments.

Unfortunately, such constructive theories were too limited to satisfy Turchin's thesis. For example, all constructive functions of real numbers are continuous; hence, discontinuous functions are excluded from that view of constructive mathematics. Working mathematicians reject this restriction. They consider a great amount of mathematical objects that cannot be represented by algorithms.

Thus mathematical constructivists met a great challenge: *How to define a wider notion of a linguistic process than algorithms, sufficient to represent mathematical objects.*

Valentin Turchin's *Cybernetic Foundation of Mathematics* (CFM) is a solution to this problem.

2 Basics of Cybernetic Foundation of Mathematics

CFM starts from the concept of constructive mathematics based on algorithms. Processes defined by algorithms are included in CFM. The whole of algorithmic and programming intuition works well in CFM. Valentin Turchin uses his favorite programming language Refal to define processes, but any other programming language will do, preferably a functional one in order to keep things as simple as necessary. For example, the main objects of modern mathematics - *sets* - are defined by processes that produce elements one by one. Thus, infinity is only *potential*: each element of an infinite set is produced at some finite moment of time, at no moment are all elements present together.

The next question is what kind of propositions about processes can be formulated and proved. Here Valentin Turchin departs from the solution adopted by the previous constructivists. He demonstrates that only two kinds of elementary propositions are needed:

1. that a given process (with given arguments) terminates at some step, and
2. that a given process never terminates.

All other statements about processes and mathematical objects can be expressed as propositions in this form, regarding processes defined by a mathematician in a chosen formal (programming) language.

This may sound strange (if not impossible) to mathematicians who studied the theory of computability and know about algorithmically undecidable problems.

Here Valentin Turchin introduces in mathematical machinery the concepts that we observe in the scientific activity in natural sciences. Traditionally, mathematics is not included in sciences. We often hear: “Mathematics and sciences”. So one could that say Turchin has returned mathematics to the natural sciences.

Scientists operate well with statements they do not know in advance to be true: they propose testable *hypotheses, predictions,* whose characteristic feature is that they can be either confirmed or falsified at some moment in time; they hold the prediction true until falsified; if it is falsified they reconsider and state the negation of the prediction then propose the next hypotheses. Scientists *believe* there exists a trajectory without backtracking, a *process* of gradually increasing knowledge. This belief with respect to linguistic processes is the background of CFM.

The elementary propositions are predictions: The proposition that a given process terminates, cannot be confirmed while the process is running; but when it stops, the proposition is confirmed and becomes certainly true. The proposition that a process does not terminate can never be confirmed, but if it does it is falsified and mathematicians return back and add its negation (saying that the process terminates) to their knowledge.

When we explain the meaning of the propositions, predictions, and truths by referring to the notion of mathematicians, their knowledge, their proposing of predictions, and their process of gaining knowledge by confirmations and falsifications of the predictions, we speak about a *model of a mathematician.* This model is introduced into the mathematical machinery and becomes one of the first-class processes considered in CFM. The previous philosophy of mathematics assumes mathematics is *objective* in the sense that while doing research, mathematicians are outside of mathematics, they are merely observers of eternal mathematical phenomena, observers who do not influence the mathematical machinery but focus their attention on interesting phenomena to discover truths and proofs about them.

Valentin Turchin said that he had introduced mathematicians to mathematics in the same way as modern physics introduced physicists, as observers, to physical theories.

This sounds fantastic. Nevertheless Valentin Turchin has achieved this ambitious goal. In his texts [1–4] are many definitions of mathematical objects including the model of the classic set theory, in terms which modern mathematicians like to define their objects. Hence, he has proved that Turchin’s thesis is met at least for objects expressible in set theory.

3 Mechanical and Metamechanical Process

In CFM, proper algorithms are referred to as *mechanical processes.* Processes of general form defined in the programming language of CFM theory can additionally access primitive *cognitive functions* that answer questions like “Does this process terminates or not?”:

– if *process p terminates* then … else …
– if *process p never terminates* then … else …

If the answer is not yet confirmed or falsified for the respective branch of the conditional and we have no reason to consider it contradictory (this notion is also defined in CFM), a truth value can be assigned to it as a prediction.

Processes that use predictions as predicates are referred to as *metamechanical processes.* The collection of predictions that are available at a given moment of time is kept as the *current knowledge of (the model of) the mathematician.* It is considered as an infinite process producing all true predictions like other infinite processes.

Valentin Turchin demonstrated how all essential mathematical notions are defined as metamechanical processes and we, the authors of mathematical theories, should reason about them. Specific mathematical theories can introduce additional cognitive functions in the same style. For example, Valentin Turchin interpreted set theory with the use of the second cognitive primitive: the process that enumerates all sentences that define sets, that is, represents the universe of sets. (Those who remember the famous set-theoretical paradoxes should immediately conclude that this process cannot be a definition of some set and cannot be produced as an element of this collection.)

Reasoning about metamechanical processes is nontrivial if possible at all. Let us consider the most subtle point to lift the veil off CFM a little.

4 Objective Interpretability

Having introduced (the model of) mathematicians into the mathematical machinery by allowing access to their knowledge, we met (the model of) of free will: if the behavior of mathematicians and content of their knowledge are deterministic, than there is nothing essentially new in the notion of a metamechanical process. Many kinds of such generalizations of algorithms have already been considered in mathematics and found to be incomplete. Hence, we must expect that a complete (w.r.t. Turchin's thesis) notion of metamechanical processes inevitably allows us to define *non-deterministic* processes as well. This means that in the hands of one (model of the) mathematician one result may be produced (e.g., some process terminated), while in the hands of another (model of the) mathematician a different result is returned (e.g., the process with the same definition did not terminate).

This is the next strange thing of CFM, considered unacceptable by classic mathematicians. Valentin Turchin did not explain how to deal with non-deterministic processes in general. However, as far as the interpretation of classic mathematics, which all mathematicians believe to be deterministic, is concerned, it is natural to expect that the metamechanical processes used to define classic mathematical notions are deterministic.

In CFM, processes that produce the same results in hands of different instantiations of the model of the mathematician (that is with different initial true knowledge) are referred to as *objectively interpretable* metamechanical processes.

Thus, we, real mathematicians (not models), should use the CFM machinery in the following way:

1. Define some notions of a mathematical theory under study as processes in the programming language of CFM.
2. Prove somehow (in meta-theory, formally or informally, *etc.*) that the defined processes are objectively interpretable.
3. Then freely use these definitions in our mathematical activity and, in particular, in the definitions of next notions and learning truths about these processes.

One may ask: What is the reliability of (meta-) proofs in Item 2? — The answer is: It depends. It must be sufficient for your activities in Item 3. It is the choice of your free will.

Notice that not all definitions are subject to non-trivial proofs in Item 2. There is no problem with new definitions that do not call directly cognitive functions of the model of the mathematician and use only functions that are already proven objectively interpretable. Such definitions are automatically objectively interpretable as well.

5 Interpretation of Set Theory

Valentin Turchin has demonstrated how to define metamechanical processes and prove they are objectively interpretable for the classic mathematical logic and set theory. He considered in turn all axioms of the classic logic and Zermelo–Fraenkel set theory, gave definitions of the corresponding processes and proved their objective interpretability.

The proof techniques ranged from quite trivial, based on programmers' intuition, to rather sophisticated. Almost all Zermel–Fraenkel axioms are existential: they state existence of certain mathematical objects in set theory. Each existential statement is interpreted in the constructive style: by the definition of the process that meets the property formulated in the axiom. For example, the axiom of existence of the empty set is a process that returns the empty list in one step; the axiom of existence of the pair of two arbitrary sets is a process that produces the list of the two elements.

It is intriguing that Valentin Turchin did not manage to find a definition of a process and a proof of its objective interpretability for the axiom schema of replacement, which was the last added by Fraenkel to form the Zermelo–Fraenkel axioms. Its interpretation in CFM remains an open problem.

6 Open Problems

Valentin Turchin laid the foundations for new constructive mathematics. However, these are only the first steps and much remains to do to turn the new foundation of mathematics into a new paradigm. We list some evident open problems to provoke the reader:

- Complete Valentin Turchin's interpretation of the Zermelo–Fraenkel set theory. He defined the corresponding process for a particular case only, where an objectively interpretable function is given in the antecedent of the axiom rather than an objectively interpretable relation.

- Are there any interesting applications of non-deterministic, non-objectively interpretable processes? If yes, how can we apply them in our mathematical activity. CFM does not prohibit this, but Valentin Turchin did not suggest a way to deal with such definitions. He only demonstrated how to exclude them from our discourse.
- Formalize the way of Turchin's reasoning where he proved objective interpretability, in any modern mathematical theory of your choice. Find interesting applications where such formalization is difficult.
- Where does the CFM style of reasoning and thinking do better than the classical one? Find interesting applications where CFM may clarify or explain something better than can classic mathematics and the previous constructivism.
- Examine the correspondence of CFM with existing constructive theories. For example, how to compare CFM, without types and with a language of CFM statically untyped, with the dependent types theory and programming of proofs in a typed functional programming language.
- Develop a proof assistant for CFM, which will help us work with constructive definitions of metamechanical processes and check proofs of objective interpretability.
- It is fair to predict that supercompilation will become an important tool for the manipulation of mathematical definitions in CFM, and extraction and proofs of their properties. Will supercompilers be somehow extended to become effectively applicable to CFM code? Develop and implement a supercompiler to be used as a CFM proof assistant.

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